BIHARMONIC CONFORMAL IMMERSIONS INTO 3-DIMENSIONAL MANIFOLDS

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Abstract

Motivated by the beautiful theory and the rich applications of harmonic conformal immersions and conformal immersions of constant mean curvature (CMC) surfaces, we study biharmonic conformal immersions of surfaces into a generic 3-manifold. We first derive an invariant equation for such immersions, we then try to answer the question, "what surfaces can be biharmonically conformally immersed into Euclidean 3-space $\mathbb{R}^3$?" We prove that a circular cylinder is the only CMC surface that can be biharmonically conformally immersed into $\mathbb{R}^3$; We obtain a classification of biharmonic conformal immersions of complete CMC surfaces into $\mathbb{R}^3$ and hyperbolic 3-spaces. We also study rotational surfaces that can be biharmonically conformally immersed into $\mathbb{R}^3$ and prove that a circular cone can never be biharmonically conformally immersed into $\mathbb{R}^3$.

1. Introduction

In this paper, all manifolds, maps, vector fields are assumed to be smooth and Einstein summation convention is used.

A biharmonic map is a map $\varphi : (M, g) \longrightarrow (N, h)$ between Riemannian manifolds that is a critical point of the bienergy functional

$$E_2(\varphi, \Omega) = \frac{1}{2}\int_{\Omega} |\tau(\varphi)|^2 \,\mathrm{d}x$$

for every compact subset Ω of M, where $\tau(\varphi) = \mathrm{Trace}_g \nabla \mathrm{d}\varphi$ is the tension field of φ vanishing of which means φ is a harmonic map. Biharmonic map equation

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is the Euler-Lagrange equation of this functional which can be written as ([Ji1])

$$\tau^2(\varphi) := \mathrm{Trace}_g(\nabla^{\varphi}\nabla^{\varphi} - \nabla^{\varphi}_{\nabla^M})\tau(\varphi) - \mathrm{Trace}_g R^N(\mathrm{d}\varphi, \tau(\varphi))\mathrm{d}\varphi = 0,$$

where R^N denotes the curvature operator of (N, h) with the convention

$$R^N(X,Y)Z = [\nabla^N_X, \nabla^N_Y]Z - \nabla^N_{[X,Y]}Z.$$

Biharmonic submanifolds are referred to those submanifolds whose defining isometric immersions are giving by biharmonic maps. The notion of biharmonic maps is a natural generalization of that of harmonic maps and biharmonic submanifolds include minimal submanifolds as a subclass. It is well known that harmonic conformal immersions of surfaces are exactly conformal minimal immersions of surfaces which have been in the focus of study for many decades and the rich theory of which has exhibited a beautiful interplay among geometry, topology and complex analysis (See [CM], [Co], [Ke] and the references therein).

One motivation of this paper is to explore how far the beautiful theory, useful techniques, and important applications of the minimal surfaces can go in the following direction of generalization:

$$\{\text{Minimal surfaces in } \mathbb{R}^3\} = \{\text{Harmonic conformal immersions : } \mathrm{M}^2 \longrightarrow \mathbb{R}^3\}$$

$$\subset$$

$$\{\text{Biharmonic conformal immersions : } \mathrm{M}^2 \longrightarrow \mathbb{R}^3\}$$

We say a hypersurface in a Riemannian manifold (N^{m+1}, h) defined by an isometric immersion $\varphi : (M^m, \bar{g}) \longrightarrow (N^{m+1}, h)$ can be biharmonically conformally immersed into (N^{m+1}, h), if there exists a function $\lambda : M^m \longrightarrow \mathbb{R}^+$ such that the conformal immersion $\varphi : (M^m, \lambda^{-2}\bar{g}) \longrightarrow (N^{m+1}, h)$ is a biharmonic map.

In this paper, we study biharmonic conformal immersions of surfaces into a generic 3-dimensional Riemannian manifold. After deriving an invariant equation for such immersions, we attempt to answer the question, "what surfaces can be biharmonically conformally immersed into Euclidean 3-space $\mathbb{R}^3$?" Among other things, we prove that a circular cylinder is the only CMC surface that can be biharmonically conformally immersed into $\mathbb{R}^3$; We obtain a classification of biharmonic conformal immersions of complete CMC surfaces into $\mathbb{R}^3$ and hyperbolic 3-spaces. For non-constant mean curvature surfaces, we obtain conditions for rotational surfaces that can be biharmonically conformally immersed into $\mathbb{R}^3$

and prove that a circular cone can never be biharmonically conformally immersed into $\mathbb{R}^3$.

2. Conformal biharmonic hypersurfaces

Biharmonic hypersurfaces in a generic Riemannian manifold were studied in [Ou1] where, among other things, the following theorem was proved.

Theorem 2.1. ([Ou1]) *Let $\varphi : M^m \longrightarrow N^{m+1}$ be an isometric immersion of codimension-one with mean curvature vector $\eta = H\xi$. Then φ is biharmonic if and only if:*

$$\begin{cases}\Delta H - H|A|^2 + H\mathrm{Ric}^N(\xi,\xi) = 0,\\ 2A\,(\mathrm{grad}\, H) + \frac{m}{2}\mathrm{grad}\, H^2 - 2\,H\,(\mathrm{Ric}^N\,(\xi))^{\top} = 0,\end{cases}$$

where $\mathrm{Ric}^N : T_qN \longrightarrow T_qN$ denotes the Ricci operator of the ambient space defined by $\langle \mathrm{Ric}^N\,(Z), W\rangle = \mathrm{Ric}^N(Z,W)$, A is the shape operator of the hypersurface with respect to the unit normal vector ξ, and Δ and grad *denote the Laplace and the gradient operators defined by the induced metric on the hypersurface.*

Definition 2.2. *A hypersurface in a Riemannian manifold (N^{m+1}, h) defined by an isometric immersion $\varphi : (M^m, \bar{g}) \longrightarrow (N^{m+1}, h)$ is said to admit a* **biharmonic conformal immersion** *into (N^{m+1}, h), if there exists a function $\lambda : M^m \longrightarrow \mathbb{R}^+$ such that the conformal immersion $\varphi : (M^m, \lambda^{-2}\bar{g}) \longrightarrow (N^{m+1}, h)$ with conformal factor λ is a biharmonic map. In such a case, we also say that the hypersurface $\varphi : (M^m, \bar{g}) \longrightarrow (N^{m+1}, h)$ can be* **biharmonically conformally immersed** *into (N^{m+1}, h)*

Remark 1. (1) Clearly, for every biharmonic conformal immersion $\varphi : (M^m, \bar{g}) \longrightarrow (N^{m+1}, h)$ with $\varphi^*h = \lambda^2\bar{g}$, the associated hypersurface $\varphi : (M^m, \varphi^*h) \longrightarrow (N^{m+1}, h)$ admits biharmonic conformal immersion into (N^{m+1}, h) since $\lambda^{-2}(\varphi^*h) = \bar{g}$.
(2) It is well known that a surface is minimal if and only if its defining isometric immersion $\varphi : M^2 \longrightarrow (N^n, h)$ is harmonic. Since harmonicity of a map from a surface is invariant under conformal changes of the metric on the surface we conclude that for any positive function λ, the conformal immersion $\varphi : (M^2, \lambda^{-2}\varphi^*h) \longrightarrow (N^n, h)$ is again harmonic and hence trivially biharmonic. It follows from Definition 2.2 that a minimal surface $\varphi : M^2 \longrightarrow (N^n, h)$ can always be *trivially* biharmonically conformally immersed in (N^n, h). For this reason, in the rest of this paper, "biharmonic conformal immersions of surfaces" will always mean biharmonic conformal immersions of **non-minimal** surfaces.

Proposition 2.3. *A hypersurface $\varphi : (M^m, g) \longrightarrow (N^{m+1}, h)$ with mean curvature vector $\eta = H\xi$ with respect to the unit normal vector field ξ can be biharmonically conformally immersed into (N^{m+1}, h) if and only if there exists a function $\lambda : M \longrightarrow (0, \infty)$ such that*

$$\begin{aligned}\lambda^4\tau^2(\varphi, g) &= -(m-2)J^{\varphi}_{\bar g}(\mathrm{d}\varphi(\mathrm{grad}_{\bar g}\,\ln\lambda)) + 2m\lambda^2(-\Delta_{\bar g}\ln\lambda - 2\left|\mathrm{grad}_{\bar g}\,\ln\lambda\right|^2_{\bar g})\eta\\ &\quad +m(m-6)\lambda^2\nabla^{\varphi}_{\mathrm{grad}_{\bar g}\,\ln\lambda}\,\eta. \qquad (1)\end{aligned}$$

Proof. Let $\bar g = \lambda^{-2}g$, one can easily see that map $\varphi : (M^m, \bar g) \longrightarrow (N^{m+1}, h)$ becomes a conformal immersion since $\varphi^* h = \lambda^2\bar g$. The proposition then follows from Proposition 1 in [Ou2]. □

Theorem 2.4. *A conformal immersion*

$$\varphi : (M^2, \bar g) \longrightarrow (N^3, h) \qquad (2)$$

into a 3-dimensional manifold with $\varphi^*h = \lambda^2\bar g$ is biharmonic if and only if

$$\begin{cases}\Delta H - H[|A|^2 - \mathrm{Ric}^N(\xi,\xi) - \lambda^{-2}\Delta(\lambda^2)] + 4g(\mathrm{grad}\,\ln\lambda, \mathrm{grad}H) = 0,\\ A(\mathrm{grad}H) + H[\mathrm{grad}H - \,(\mathrm{Ric}^N\,(\xi))^{\top} + 2\,A(\mathrm{grad}\,\ln\lambda)] = 0\end{cases} \qquad (3)$$

where ξ, A, and H are the unit normal vector field, the shape operator, and the mean curvature function of the surface $\varphi(M) \subset (N^3, h)$ respectively, and the operators Δ, grad and $|,|$ are taken with respect to the induced metric $g = \varphi^*h = \lambda^2\bar g$ on the surface.

Proof. Let $\varphi : (M^2, \varphi^*h = \lambda^2\bar g) \longrightarrow (N^3, h)$ be the isometric immersion associated to the conformal immersion (2). Then, the conformal immersion (2) is biharmonic if and only if the associated surface can be biharmonically conformally immersed into (N^3, h) since $\lambda^{-2}(\varphi^*h) = \bar g$. It follows from (1) with $m = 2$ that the conformal immersion φ is biharmonic if and only if

$$\lambda^2\tau^2(\varphi, g) = -4(\Delta_{\bar g}\ln\lambda + 2\left|\mathrm{grad}_{\bar g}\,\ln\lambda\right|^2_{\bar g})\eta - 8\nabla^{\varphi}_{\mathrm{grad}_{\bar g}\,\ln\lambda}\,\eta, \qquad (4)$$

where $\tau^2(\varphi, g)$ denotes the bitension field of the associated isometric immersion $\varphi : (M^2, g = \lambda^2\bar g) \longrightarrow (N^3, h)$ with mean curvature vector $\eta = H\xi$, where ξ and H are the unit normal vector field and the mean curvature function of the surface $\varphi(M)$ respectively. Using the formula for $\tau^2(\varphi, g)$ given in [Ou1] with $m = 2$, we have

$$\begin{aligned}\tau^2(\varphi, g) &= 2\Big[\Delta H - H\,|A|^2 + H\mathrm{Ric}^N(\xi,\xi)\Big]\xi\\ &\quad - 2\Big[2A(\mathrm{grad}H) + \mathrm{grad}(H^2) - 2\,H\,(\mathrm{Ric}\,(\xi))^{\top}\Big].\end{aligned}$$

Substituting this into (4) we have

$$\begin{aligned}
(5)\qquad & \lambda^2\big[\Delta H - H\,|A|^2 + H\mathrm{Ric}^N(\xi,\xi)\big]\xi \\
& -\lambda^2\Big[2A(\mathrm{grad}H) + \mathrm{grad}(H^2) - 2\,H\,(\mathrm{Ric}\,(\xi))^{\top}\Big] \\
= \;& -2(\Delta_{\bar g}\ln\lambda + 2\big|\mathrm{grad}_{\bar g}\ln\lambda\big|_{\bar g}^{2})\eta - 4\nabla^{\varphi}_{\mathrm{grad}_{\bar g}\ln\lambda}H\xi.
\end{aligned}$$

On the other hand, it is easy to check that the transformations of Laplacian and the gradient operators under a conformal change of metrics $g = \lambda^2\bar g$ on a two-dimensional manifold are given by

$$(6)\qquad \Delta_{\bar g}u = \lambda^2\Delta u, \quad \mathrm{grad}_{\bar g}u = \lambda^2\mathrm{grad}u.$$

Using these we have

$$\begin{aligned}
(7)\qquad -4\nabla^{\varphi}_{\mathrm{grad}_{\bar g}\ln\lambda}H\xi &= -4\bar g(\mathrm{grad}_{\bar g}\ln\lambda, \mathrm{grad}_{\bar g}H)\xi + 4H\,A(\mathrm{grad}_{\bar g}\ln\lambda) \\
&= -4\lambda^2 g(\mathrm{grad}\ln\lambda, \mathrm{grad}H)\xi + 4\lambda^2 H\,A(\mathrm{grad}\ln\lambda)
\end{aligned}$$

Substituting (6) and (7) into (5) yields

$$\begin{aligned}
& \big[\Delta H - H\,|A|^2 + H\mathrm{Ric}^N(\xi,\xi)\big]\xi \\
& -[2A(\mathrm{grad}H) + \mathrm{grad}(H^2) - 2\,H\,(\mathrm{Ric}\,(\xi))^{\top}] \\
= \;& -\lambda^{-2}(\Delta\lambda^2)H\xi - 4g(\mathrm{grad}\ln\lambda, \mathrm{grad}H)\xi + 4H\,A(\mathrm{grad}\ln\lambda).
\end{aligned}$$

By comparing the tangential and normal parts of this vector equation we obtain the theorem. □

Corollary 2.5. *A conformal immersion $\varphi : (M^2, \bar g) \longrightarrow (N^3(C), h_0)$ into 3-dimensional space of constant sectional curvature C with $\varphi^* h_0 = \lambda^2\bar g$ is biharmonic if and only if*

$$(8)\qquad \begin{cases} \Delta H - H[|A|^2 - 2C - \lambda^{-2}\Delta(\lambda^2)] + 4g(\mathrm{grad}\ln\lambda, \mathrm{grad}H) = 0, \\ A(\mathrm{grad}H) + H[\mathrm{grad}H + 2\,A(\mathrm{grad}\ln\lambda)] = 0, \end{cases}$$

where ξ is the unit normal vector field of the surface $\varphi(M) \subset \mathbb{R}^3$ and A and H are the shape operator and the mean curvature function of the surface respectively, and the operators Δ, grad and $|,|$ are taken with respect to the induced metric $g = \varphi^* h = \lambda^2\bar g$ on the surface.

Proof. This follows from Theorem 2.4 and the fact that a 3-dimensional space of constant sectional curvature C is an Einstein manifold with $\mathrm{Ric}^N(\xi,\xi) = 2C$ and $\mathrm{Ric}^N\,(\xi))^{\top} = 0$. □

Remark 2. When $C = 0$, Corollary 2.5 recovers Theorem 2 in [Ou2] where the notation $|A|^2$ denotes the norm square of the second fundamental form taken with respect to the conformal metric $\bar g$ rather than the induced metric.

Using Theorem 2.4 and Definition 2.2 we have

Corollary 2.6. *A surface $\varphi : M^2 \longrightarrow (N^3, h)$ with the induced metric $g = \varphi^* h$, the shape operator A, and the mean curvature function H can be biharmonically conformally immersed into (N^3, h) if and only if there exists a positive function λ defined on M^2 that solves Equation (3) .*

For CMC surfaces, we have

Corollary 2.7. *A non-zero constant mean curvature surface $\varphi : M^2 \longrightarrow (N^3, h)$ with the induced metric $g = \varphi^* h$ and the shape operator A can be harmonically conformally immersed into (N^3, h) if and only if and only if there exists a positive function λ defined on M^2 such that*

$$
\begin{cases} \Delta(\lambda^2) = \lambda^2[|A|^2 - {\rm Ric}^N(\xi, \xi)], \\ A({\rm grad}\ {\rm ln}\lambda) = \frac{1}{2}({\rm Ric}^N\,(\xi))^{\top} \end{cases} \tag{9}
$$

It is well known (see e.g., [CI], [Ji2]) that if a surface admits a biharmonic homothetic immersion into Euclidean 3-space $\mathbb{R}^3$, then it has to be a minimal surface. It was proved in [Ou2] that a circular cylinder in $\mathbb{R}^3$ can be biharmonically conformally immersed into $\mathbb{R}^3$. As to the general question: what surfaces in $\mathbb{R}^3$ can be biharmonically conformally immersed into $\mathbb{R}^3$, we have

Corollary 2.8. *A surface $M^2 \longrightarrow \mathbb{R}^3$ with mean curvature function H and the shape operator A can be biharmonically conformally immersed into $\mathbb{R}^3$ if and only if there exists a positive function λ defined on M^2 such that*

$$
\begin{cases} \Delta H - H[|A|^2 - \lambda^{-2}\Delta(\lambda^2)] + 4g({\rm grad}\ {\rm ln}\lambda, {\rm grad}H) = 0, \\ A({\rm grad}H) + H[{\rm grad}H + 2\,A({\rm grad}\ {\rm ln}\lambda)] = 0, \end{cases} \tag{10}
$$

where the Laplace operator Δ, grad *and $|,|$ are taken with respect to the induced metric $g = \varphi^* h_0$ on M.*

In particular, a surface M^2 in $\mathbb{R}^3$ of constant mean curvature $H \neq 0$ can be biharmonically conformally immersed into $\mathbb{R}^3$ if and only if there exists a positive function λ defined on M^2 such that

$$
\begin{cases} -|A|^2 + \lambda^{-2}\Delta\lambda^2 = 0, \\ A({\rm grad}\ {\rm ln}\lambda) = 0. \end{cases} \tag{11}
$$

By the nature of the PDE (10), the question what surfaces can be biharmonically conformally immersed into $\mathbb{R}^3$ does not seem to be a simple one. Even in the family of the CMC surfaces, we know that a plane (being totally geodesic and hence minimal) always admits trivial biharmonic conformal immersions, we

also know ([Ou2]) that a circular cylinder can be biharmonically conformally immersed into $\mathbb{R}^3$. However, as the following corollary shows, not all CMC surfaces enjoy this property.

Corollary 2.9. *No part of the standard sphere S^2 can be biharmonically conformally immersed into $\mathbb{R}^3$.*

Proof. One can parametrize a piece of the unit sphere as $\phi : D \longrightarrow \mathbb{R}^3$, with $\phi(u,v) = (\cos u \cos v, \cos u \sin v, \sin u)$, then, a straightforward computation yields the shape operator as

$$A = \begin{pmatrix} 1 & 0 \\ 0 & \cos^2 u \end{pmatrix}. \tag{12}$$

If there were a biharmonic conformal immersion of the unit sphere into $\mathbb{R}^3$. Then, Equation (12), together with the second equation of (11), would imply that grad $\ln\lambda = 0$, and hence $\lambda =$ constant. It follows from this and the first equation of (11) that $|A|^2 = 0$, which is clearly a contradiction to (12). Thus, we obtain the corollary. □

Our next theorem shows that circular cylinders are the only constant mean curvature surfaces that admit biharmonic conformal immersions into $\mathbb{R}^3$. To prove the theorem, we will need the following theorem which gives the existence of special isothermal coordinates on a nonzero constant mean curvature surface and the evidence of the existence of many CMC surfaces in $\mathbb{R}^3$.

Theorem A (see, e.g., [Ke], p.22): Let $M^2 \longrightarrow \mathbb{R}^3$ be a surface of nonzero constant mean curvature H, and let $p \in M$ be a non-umbilical point. Then there exist isothermal coordinates (u, v) in a neighborhood of p satisfying

$$I = \frac{e^{2w}}{2H}(du^2 + dv^2), \tag{13}$$

$$II = e^w \cosh w \, du^2 + e^w \sinh w \, dv^2, \tag{14}$$

where w=w(u,v) is a solution of the sinh-Gordon equation

$$w_{uu} + w_{vv} + 2H \cosh w \sinh w = 0.$$

Conversely, for any given positive constant H and a solution w of sinh-Gordon equation, there exists a CMC surface whose first and the second fundamental forms are given by (13), and (14).

Now, we are ready to prove the following theorem.

Theorem 2.10. *A constant mean curvature surface can be biharmonically conformally immersed into $\mathbb{R}^3$ if and only if it is a part of a plane or a circular cylinder.*

Proof. Let $\varphi : (M^2, \varphi^* h_0) \longrightarrow (\mathbb{R}^3, h_0)$ be a surface of constant mean curvature H. If the surface is totally umbilical, then it is well known that it is a part of a plane or a sphere. As we mentioned in the paragraph preceding Corollary 2.9 that a plane can always be (trivially) biharmonically conformally immersed in $\mathbb{R}^3$. On the other hand, we know from Corollary 2.9 that no part of a sphere can be biharmonically conformally immersed into $\mathbb{R}^3$.

If the surface is not totally umbilical, then, by Theorem A, we can choose local isothermal coordinates (u, v) so that its first and second fundamental forms are given by (13) and (14). Thus, the coefficients of the first and the second fundamental forms are given by

$$g_{11} = g_{22} = \frac{e^{2w}}{2H},\ \ g_{12} = g_{21} = 0;\ \ \ g^{11} = g^{22} = \frac{2H}{e^{2w}},\ \ g^{12} = g^{21} = 0.$$

and

$$h_{11} = e^w \cosh w, \quad h_{12} = h_{21} = 0, \quad h_{22} = e^w \sinh w.$$

A straightforward computation yields

$$\begin{aligned} A(\partial_1) &= g^{kl} h_{l1} \partial_k = 2He^{-w} \cosh w\, \partial_1, \\ A(\partial_2) &= g^{kl} h_{l2} \partial_k = 2He^{-w} \sinh w\, \partial_2, \\ \mathrm{grad}(\ln\lambda) &= 2He^{-2w}[(\partial_1 \ln \lambda)\partial_1 + (\partial_2 \ln \lambda)\partial_2], \\ A(\mathrm{grad}(\ln\lambda)) &= 4H^2 e^{-3w}[(\partial_1 \ln \lambda) \cosh w\, \partial_1 + (\partial_2 \ln \lambda) \sinh w\, \partial_2]. \end{aligned} \tag{15}$$

Substituting (15) into the second equation of (11) we obtain

$$\begin{cases} 4H^2 e^{-3w} (\partial_1 \ln \lambda) \cosh w = 0, \\ 4H^2 e^{-3w} (\partial_2 \ln \lambda) \sinh w = 0. \end{cases} \tag{16}$$

Since $H \neq 0$ we solve (16) to have $\lambda =$ constant, or $w = 0$ and $\partial_1 \ln \lambda = 0$. In the case of $\lambda =$ constant, the biharmonic conformal immersion $\varphi : (M^2, \lambda^{-2}\varphi^* h_0) \longrightarrow (\mathbb{R}^3, h_0)$ is actually a biharmonic homothetic immersion which is minimal in $\mathbb{R}^3$ by a well known result in [CI]. This contradicts to the assumption that $H \neq 0$. In the case of $w = 0$ and $\partial_1 \ln \lambda = 0$, it is not difficult to see from (13) and (14) that the CMC surface is isometric to a circular cylinder. Conversely, we know that a plane always admits trivial biharmonic conformal immersion into $\mathbb{R}^3$. We also know from [Ou2] that a circular cylinder always admits a biharmonic conformal immersion into $\mathbb{R}^3$. Summarizing the above results we obtained the theorem. □

Our next theorem gives conditions for a rotational surface that can be biharmonically conformally immersed into $\mathbb{R}^3$.

Theorem 2.11. *A non-minimal rotational surface* $r(u,v) = (f(u)\cos v,\ f(u)\sin v,\ g(u))$ *obtained by rotating the arclength parametrized curve* $(C): x = f(u), z = g(u)$ *about* z*-axis can be biharmonically conformally immersed into* $\mathbb{R}^3$ *if and only if there exists a positive function* λ *on the surface depending only on variable* u *such that*

$$kH' + HH' + 2kH(\ln\lambda)' = 0, \tag{17}$$

$$\begin{aligned} &2(\ln\lambda)'' + 2[\ln(fH^2)]'(\ln\lambda)' + 4(\ln\lambda)'^2 = \\ &k^2 + (g'f^{-1})^2 - H''H^{-1} - f'f^{-1}H'H^{-1}, \end{aligned} \tag{18}$$

where $k = -f''g' + g''f'$ *is the curvature of the generating curve (C),* $H = [k + g'f^{-1}]/2$ *is the mean curvature of the rotational surface.*

Proof. A straightforward computation gives

$$\begin{aligned} r_u &= (f'\cos v,\ f'\sin v,\ g'), \quad r_v = (-f\sin v,\ f\cos v,\ 0),\\ r_{uu} &= (f''\cos v,\ f''\sin v,\ g''), \quad r_{uv} = (-f'\sin v,\ f'\cos v,\ 0),\\ r_{vv} &= (-f(u)\cos v,\ -f(u)\sin v,\ 0). \end{aligned}$$

The unit normal vector field of the rotational surface can be chosen to be $\xi = \frac{r_u\times r_v}{|r_u\times r_v|} = (-g'\cos v, -g'\sin v, f')$. A further computation gives the coefficients of the first fundamental form

$$g_{11} = \langle r_u, r_u\rangle = 1,\ \ g_{12} = g_{21} = \langle r_u, r_v\rangle = 0,\ \ g_{22} = \langle r_v, r_v\rangle = f^2,$$
$$\text{and hence}\ \ g^{11} = 1,\ \ g^{12} = g^{21} = 0,\ \ g^{22} = f^{-2}$$

and the second fundamental form

$$h_{11} = \langle r_{uu}, \xi\rangle = -f''g' + g''f' = k,\quad h_{12} = h_{21} = \langle r_{uv}, \xi\rangle = 0,\quad h_{22} = \langle r_{vv}, \xi\rangle = fg'.$$

Using the natural frame $\{\partial_1 = r_u, \partial_2 = r_v, \xi\}$ on $\mathbb{R}^3$ adapted to the rotational surface we compute

$$\begin{aligned}
A(\partial_1) &= g^{ij}h_{j1}\partial_i = k\partial_1,\\
A(\partial_2) &= g^{ij}h_{j2}\partial_i = g'f^{-1}\partial_2,\\
|A|^2 &= g^{kl}g^{ij}h_{ki}h_{jl} = k^2 + (g'f^{-1})^2,\\
H &= \frac{1}{2}g^{ij}h_{ij} = (k + g'f^{-1})/2,\\
\text{grad}H &= g^{kl}\partial_k(H)\partial_l = H'\partial_1, \quad (19)\\
A(\text{grad}H) &= kH'\,\partial_1, \quad (20)\\
\text{grad}(\ln\lambda) &= (\partial_1 \ln\lambda)\partial_1 + f^{-2}(\partial_2 \ln\lambda)\partial_2,\\
A(\text{grad}(\ln\lambda)) &= k(\partial_1 \ln\lambda)\partial_1 + g'f^{-3}(\partial_2 \ln\lambda)\partial_2. \quad (21)
\end{aligned}$$

Substituting (19), (20), and (21) into the second equation of (10) we obtain

$$(22)\quad \begin{cases} kH' + HH' + 2kH(\partial_1 \ln\lambda) = 0,\\ 2Hg'f^{-3}(\partial_2 \ln\lambda) = 0.\end{cases}$$

Since the rotational surface is assumed to be non-minimal, we have $H \neq 0$. This, together with the second equation of (22) implies that

$$\partial_2 \ln\lambda = 0,$$

which means λ depends only on the variable u.

In order to expand the first equation of (10) we compute

$$(23)\quad \Delta H = g^{ij}H_{ij} - g^{ij}\Gamma_{ij}^k H_k = H'' - (\Gamma_{11}^1 + \Gamma_{22}^1)H' = H'' + f'f^{-1}H',$$

where in obtaining the second equality we have used the fact that H depends only on variable u. A similar computation yields

$$(24)\quad \begin{aligned}\lambda^{-2}\Delta\lambda^2 &= 2\Delta(\ln\lambda) + 4|\text{grad}\ln\lambda|^2\\ &= 2(\ln\lambda)'' + 2f'f^{-1}(\ln\lambda)' + 4(\ln\lambda)'^2.\end{aligned}$$

Substituting (23) and (24) into the first equation of (10) and simplifying the result we obtain

$$(25)\quad \begin{aligned}&2(\ln\lambda)'' + 2[\ln(fH^2)]'(\ln\lambda)' + 4(\ln\lambda)'^2 =\\ &k^2 + (g'f^{-1})^2 - H''H^{-1} - f'f^{-1}H'H^{-1}.\end{aligned}$$

Combining (22) and (25) we obtain the theorem. □

As an application of Theorem 2.11 we have

Corollary 2.12. *A circular cone can never be biharmonically conformally immersed into* $\mathbb{R}^3$.

Proof. A circular cone is a rotational surface with the generating curve being a straight line whose curvature $k = 0$. It follows from Equation of (17) that if a circular cone admitted a biharmonic conformal immersion into $\mathbb{R}^3$, then $H' = 0$ and hence its mean curvature would be constant, which is clearly a contradiction since the mean curvature of a circular cone is not constant. □

Our next theorem gives a classification of biharmonic conformal immersions of complete constant mean curvature surfaces into $\mathbb{R}^3$.

Theorem 2.13. *Let* $\varphi : (M^2, \bar{g}) \longrightarrow (\mathbb{R}^3, h_0)$ *be a biharmonic conformal immersion of a surface into* 3*-dimensional Euclidean space with* $\varphi^* h_0 = \lambda^2 \bar{g}$ *being complete and* $\int_M \lambda^6 \, dv_{\bar{g}} < \infty$. *If the surface* $\varphi(M) \subset \mathbb{R}^3$ *has constant mean curvature, then the biharmonic conformal immersion* φ *is minimal.*

Proof. By Corollary 2.5, the equation for a biharmonic conformal immersion into $\mathbb{R}^3$ reduces to

$$\begin{cases} \Delta H - H[|A|^2 - \lambda^{-2}\Delta(\lambda^2)] + 4g(\text{grad}\ \ln\lambda, \text{grad} H) = 0, \\ A(\text{grad} H) + H[\text{grad} H + 2\, A(\text{grad}\ \ln\lambda)] = 0 \end{cases} \tag{26}$$

where A and H are the shape operator, and the mean curvature function of the surface $\varphi(M) \subset (\mathbb{R}^3, h_0)$ respectively, and the operators $\Delta,$ grad and $|,|$ are taken with respect to the induced metric $g = \varphi^* h0 = \lambda^2 \bar{g}$ on M.

If the surface $\varphi(M) \subset (\mathbb{R}^3, h_0)$ has constant mean curvature $H = 0$, then nothing is left to prove. Otherwise, if $H = \text{constant} \neq 0$, then Equation (26) reads

$$\begin{cases} \Delta\lambda^2 = |A|^2\lambda^2, \\ A(\text{grad}\ \ln\lambda) = 0. \end{cases} \tag{27}$$

Noting that the condition $\int_M \lambda^6 \, dv_{\bar{g}} < \infty$ and the first equation in (27) means that λ^2 is an L^2 solution of the Schrödinger type equation on a complete Riemannian manifold (M^2, g) we use Lemma 3.1 in [NU] to conclude that λ^2 is a constant. Thus, the biharmonic conformal immersion φ is a actually a homothetic biharmonic immersion into $\mathbb{R}^3$ which has to be a minimal immersion and hence $H = 0$, a contradiction. This completes the proof of the theorem. □

Remark 3. (1) We would like to point out that a similar argument can be used to prove that Theorem 2.13 remains true if the target space $\mathbb{R}^3$ is replaced by any 3-manifold with non-positive sectional curvature.

(2) We also point out that as the following example shows the condition $\int_M \lambda^6 dv_{\bar{g}} < \infty$ in Theorem 2.13 is sharp and cannot be dropped.

Example 1. Let $\mathbb{R}^2$ be the Euclidean plane provided with the metric $\bar{g} = e^{-y/R}(dx^2 + dy^2$. The map $\varphi : (\mathbb{R}^2, \bar{g}) \longrightarrow \mathbb{R}^3$ given by $\varphi(x, y) = (R\cos\frac{x}{R}, R\sin\frac{x}{R}, y)$ is a biharmonic conformal immersion of $\mathbb{R}^2$ into Euclidean space $\mathbb{R}^3$ with $\varphi^* h_0 = e^{y/R}\bar{g}$. The induced metric $g = e^{y/R}\bar{g}$ on $\mathbb{R}^2$ is complete and the surface $\varphi(\mathbb{R}^2) \subset \mathbb{R}^3$ is a circular cylinder which has constant mean curvature. The biharmonic conformal immersion φ is not harmonic because $\int_M \lambda^6 dv_{\bar{g}} = \infty$.

To understand the example, we notice that in this case, $\varphi_x = (-\sin\frac{x}{R}, \cos\frac{x}{R}, 0)$ and $\varphi_y = (0, 0, 1)$ so one can check that φ is a conformal immersion with $\varphi^* h_0 = dx^2 + dy^2 = \lambda^2 \bar{g}$ for $\lambda^2 = e^{y/R}$. It follows that the induced metric $g = \varphi^* h_0 = dx^2 + dy^2$ on $\mathbb{R}^2$ is the standard Euclidean metric which is complete. One can easily check that $e_1 = \varphi_x = (-\sin\frac{x}{R}, \cos\frac{x}{R}, 0)$, $e_2 = \varphi_y = (0, 0, 1)$, $\xi = (\cos\frac{x}{R}, \sin\frac{x}{R}, 0)$ form an orthonormal frame adapted to the surface. A straightforward computation yields

$$\begin{aligned}
Ae_1 &= -\frac{1}{R}e_1, \quad Ae_2 = 0, \\
H &= \frac{1}{2}(\langle Ae_1, e_1\rangle + \langle Ae_2, e_2\rangle) = -\frac{1}{2R} \neq 0 \\
|A|^2 &= \sum_{i=1}^{2} |Ae_i|^2 = \frac{1}{R^2}, \\
\mathrm{grad}(\ln\lambda) &= e_2(\ln\lambda)e_2.
\end{aligned}$$

It follows that

$$\begin{cases}
\Delta(\lambda^2) = \Delta(e^{y/R}) = \frac{\partial^2}{\partial y^2}(e^{y/R}) = \frac{1}{R^2}(e^{y/R}) = |A|^2\lambda^2, \\
A(\mathrm{grad}\ \ln\lambda) = 0,
\end{cases}$$

which means Equation (27) holds. Thus, the conformal immersion φ is indeed a biharmonic map which is not harmonic since $H \neq 0$. This does not contradict the conclusion of Theorem 2.13 because for this example the condition that $\int_M \lambda^6 dv_{\bar{g}} < \infty$ required by the theorem is not satisfied. In fact, $\int_M \lambda^6 dv_{\bar{g}} = \infty$.

In a very recent paper [NUG], Nakauchi, Urakawa and Gudmundsson prove that any biharmonic map with finite energy and finite bi-energy from a complete Riemannian manifold into a Riemannian manifold of non-positive sectional curvature has to be harmonic. We can prove the following results about biharmonic

conformal immersions which are the dual results for biharmonic horizontally conformal submersions (Theorem 4.2 and Corollary 4.3) given in [NUG].

Proposition 2.14. *Any biharmonic conformal immersion $\phi : (M^m, g) \longrightarrow (N^n, h)$ from a complete manifold into a nonpositively curved space with $\phi^* h = \lambda^2$ satisfying*

$$\int_M \lambda^2 dv_g < \infty, \text{ and}$$
$$\int_M |m\lambda^2\eta + (2-m)d\phi(\text{grad } \ln\lambda)|^2 dv_g < \infty,$$

where η denotes the mean curvature vector of the submanifold $\phi(M) \subset (N, h)$, has to be harmonic. In particular, any biharmonic conformal immersion $\phi : (M^2, g) \longrightarrow (N^n, h)$ from a complete surface into a nonpositively curved space with $\phi^* h = \lambda^2$ satisfying

$$\int_M \lambda^2 dv_g < \infty, \text{ and} \tag{28}$$
$$\int_M \lambda^4 |\eta|^2 dv_g < \infty, \tag{29}$$

where η denotes the mean curvature vector of the submanifold $\phi(M) \subset (N, h)$, has to be minimal.

Proof. For conformal immersion $\phi : (M^m, g) \longrightarrow (N^n, h)$ with $\phi^* h = \lambda^2 g$, one can easily compute its energy to have

$$E(\phi) = \int_M |d\phi|^2 dv_g = \int_M g^{ij}\phi_i^\alpha \phi_j^\beta h_{\alpha\beta} dv_g = m\int_M \lambda^2 dv_g. \tag{30}$$

On the other hand, a straightforward computation one obtains the tension field of the conformal immersion $\tau(\phi) = m\lambda^2\eta + (2-m)d\phi(\text{grad } \ln\lambda)$. It follows that the bienergy of the conformal immersion is given by

$$E_2(\phi) = \int_M |m\lambda^2\eta + (2-m)d\phi(\text{grad } \ln\lambda)|^2 dv_g. \tag{31}$$

Using Equations (30) and (31) and Theorem 2.3 in [NUG] we obtain the proposition. □

Remark 4. It is interesting to note that we can use Proposition 2.14 to conclude that a biharmonic conformal immersion $\phi : (M^2, \bar{g}) \longrightarrow \mathbb{R}^3$ of a complete CMC

surface into Euclidean 3-space with $\phi^*h=\lambda^2\bar{g}$ satisfying

$$\int_M \lambda^2 dv_{\bar{g}} \;<\; \infty, \text{ and} \tag{32}$$

$$\int_M \lambda^4 dv_{\bar{g}} \;<\; \infty \tag{33}$$

has to be minimal. On the other hand, we can use Theorem 2.13 to have the same conclusion with the assumption that $\int_M \lambda^6 dv_{\bar{g}}<\infty$. We also know that in general conditions (32) and (33) do not imply $\int_M \lambda^6 dv_{\bar{g}}<\infty$ as L^2 space is not closed under multiplication.

To finish the paper, we give a result that shows the vertical cylinders in $S^2\times\mathbb{R}$ admit biharmonic conformal immersions into $S^2\times\mathbb{R}$, a 3-manifold of nonconstant sectional curvature.

Proposition 2.15. *Let $\alpha: I\longrightarrow (S^2,h)$ be (a part of) a circle in S^2 with radius $1/k$, $(k>1)$ and let $\Sigma=\bigcup_{t\in I}\pi^{-1}(\alpha(t))$ denote the vertical cylinder in $S^2\times\mathbb{R}$. Then, the conformal immersion $\varphi:(\Sigma,\bar{g}=\lambda^{-2}\varphi^*h)\longrightarrow(N^3,h)$ is biharmonic if and only if $\lambda^2=\big(C_2e^{\pm z/R}-C_1C_2^{-1}R^2e^{\mp z/R}\big)/2$, where $R=1/\sqrt{k^2-1}$.*

Proof. We know as in [Ou1] that the cylinder has constant mean curvature $H=k/2$, and an adapted orthonormal frame $\{X,V,\xi\}$ with ξ being normal to the cylinder. The shape operator and the second fundamental form b with respect to the orthonormal frame are given by (see [Ou1])

$$\begin{aligned}
&A(X)=-\langle\nabla_X\xi,X\rangle X-\langle\nabla_X\xi,V\rangle V=kX,\\
&A(V)=-\langle\nabla_V\xi,X\rangle X-\langle\nabla_V\xi,V\rangle V=-\tau X=0;\\
&b(X,X)=\langle A(X),X\rangle=k,\;\; b(X,V)=\langle A(X),V\rangle=-\tau=0,\\
&b(V,X)=\langle A(V),X\rangle=-\tau=0,\;\; b(V,V)=\langle A(V),V\rangle=0.
\end{aligned}$$

A further computation gives

$$\begin{aligned}
&H=\frac{1}{2}(b(X,X)+b(V,V))=\frac{k}{2},\\
&A(\mathrm{grad}\,H)=A(X(\frac{k}{2})X+V(\frac{k}{2})V)=X(\frac{k}{2})A(X)=\frac{k'}{2}(kX-\tau V)=0;\\
&\Delta H=XX(H)-(\nabla_X X)H+VV(H)-(\nabla_V V)H=\frac{k''}{2}=0;\\
&|A|^2=(b(X,X))^2+(b(X,V))^2+(b(V,X))^2+(b(V,V))^2=k^2.
\end{aligned}$$

We also have [Ou1]

$$\begin{cases} \mathrm{Ric}(\xi,\xi) = \mathrm{Ric}(y'E_1 - x'E_2, y'E_1 - x'E_2) = y'^2 + x'^2 = 1, \\ \mathrm{Ric}(\xi, X) = \mathrm{Ric}(y'E_1 - x'E_2, x'E_1 + y'E_2) = x'y' - y'x' = 0, \\ \mathrm{Ric}(\xi, V) = \mathrm{Ric}(y'E_1 - x'E_2, E_3) = 0. \end{cases}$$

Substituting these into (9) we have

$$\begin{cases} -(k^2-1) + 2[\Delta \ln\lambda + 2\,|\mathrm{grad}\ln\lambda|^2] = 0, \\ A(\mathrm{grad}\ \ln\lambda) = 0. \end{cases} \tag{34}$$

Noting that the cylinder can be parametrized by $\phi(s,z) = (\alpha(s), z) \subset S^2 \times \mathbb{R}$ and that $A(\mathrm{grad}\ \ln\lambda) = X(\ln\lambda) = 0$, where X is tangent to α, we conclude that $\lambda(s,z)$ depends only on z. It follows that Equation (34) is equivalent to

$$\begin{cases} (\ln\lambda)'' + 2(\ln\lambda)'^2 = (k^2-1)/2, \\ \lambda(s,z) = \lambda(z) \end{cases}$$

or,

$$(\lambda^2)'' = \frac{1}{R^2}\lambda^2,$$

where, $R = 1/\sqrt{k^2-1}$. It follows that λ^2 is a solution of the ordinary differential equation

$$y'' = \frac{1}{R^2}y,$$

which has (see e.g., [Cu]) the first integral

$$y'^2 = y^2/R^2 + C_1. \tag{35}$$

Solving Equation (35) we have

$$y = \left(C_2 e^{\pm z/R} - C_1 C_2^{-1} R^2 e^{\mp z/R}\right)/2.$$

From this we have

$$\lambda^2 = \left(C_2 e^{\pm\sqrt{k^2-1}\,z} - C_1[C_2(k^2-1)]^{-1} e^{\mp\sqrt{k^2-1}\,z}\right)/2.$$

Thus, we obtain the proposition. □

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